SOLUTION OF LAPLACE'S EQUATION IN TOROIDAL COÖRDINATES DEDUCED BY A METHOD OF IMAGINARY INVERSION.

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This paper is based on a well known theorem of Lord Kelvin* which shows that, if $V\,(xyz)$ is a solution of Laplace's Equation,

$$\frac{1}{\rho} V \left[\frac{kx}{\rho^2} \frac{ky}{\rho^2} \frac{kz}{\rho^2} \right] \tag{1}$$

is also a solution of Laplace's Equation. Here space has been subjected to inversion with regard to a sphere of radius $\sqrt{k}$ described about the origin as centre. In (1) ρ is the distance from the origin to the point (xyz).

This theorem is usually applied in the case of real inversion, but in this paper the inversion will be imaginary, while the real results depend upon the fact that a certain cone system becomes a toroidal system by inversion.

The problem proposed is to deduce solutions of Laplace's Equation, in terms of curvilinear coördinates corresponding to a toroidal system, from a certain class of solutions (see equation (5)) in terms of polar coördinates. The results obtained are not new, and their uses, being well known, need not concern us here. The interest lies in the method used which brings out in an elementary manner the connection between two problems usually treated as quite distinct.

The equations of the cone system to be employed are

$$x^2 + y^2 = z^2 \tan^2 \theta \tag{2}$$

$$x^2 + y^2 + z^2 = r^2 \tag{3}$$

$$y = x \tan \varphi \tag{4}$$

These surfaces correspond to the ordinary system of polar coördinates for which V of Laplace's Equation may be written

$$V = [A_1 r^m + B_1 r^{-(m+1)}][A_2 \cos n\varphi + B_2 \sin n\varphi] \,.$$

$$[A_3 P_m^n (\cos \theta) + B_3 Q_m^n (\cos \theta)] \,. \tag{5}$$

The next point to be considered is the transformation which will change the system of cones, spheres, and planes into the toroidal system of surfaces.

* See, for instance, Leçons sur L'Électricité et Le Magnétisme. Pierre Duhem, Paris, 1891, Tome I, Ch. VIII.

Since both systems contain a family of meridian planes, it is desirable to preserve the axis of the planes of the cone system, and thus have one family of planes pass into the other. It follows that the centre of inversion must be at a point on the axis of z.

The transformation which will leave the meridian planes unchanged may be written in detail as follows:

$$x = x_1 \quad y = y_1 \quad z = z_1 + a \tag{6}$$

$$x_1 = \frac{kx_2}{x_2^2 + y_2^2 + z_2^2} \quad y_1 = \frac{ky_2}{x_2^2 + y_2^2 + z_2^2} \quad z_1 = \frac{kz_2}{x_2^2 + y_2^2 + z_2^2} \tag{7}$$

$$x_2 = x_3 \quad y_2 = y_3 \quad z_2 = z_3 + b \tag{8}$$

The constants a, b, k are not restricted at present.

As the result of the transformation of the family of spheres, equation (3) takes the form, after dropping subscripts,

$$(a^2 - r^2)\,(x^2 + y^2 + z^2 + 2bz + b^2) + 2azk + 2abk + k^2 = 0 \tag{9}$$

Equation (9) represents a family of spheres whose common circle of intersection is in the plane

$$2az + 2ab + k = 0\,. \tag{10}$$

In order that this plane may become the xy plane, the condition is imposed

$$2ab + k = 0\,. \tag{11}$$

Equation (9) now becomes

$$x^2 + y^2 + z^2 + b^2 = -\,2bz\left[\frac{4b^2\,r^2 + k^2}{4b^2\,r^2 - k^2}\right]. \tag{12}$$

If β be the angle between any sphere of equation (12) and the xy plane at their intersection it can be shown that

$$\cot\beta = \pm\, i\left[\frac{4b^2\,r^2 + k^2}{4b^2\,r^2 - k^2}\right]. \tag{13}$$

The fact that angles are unchanged by inversion shows that β is also the angle between the two corresponding spheres of equation (3). Finally, equation (12) has the form

$$x^2 + y^2 + z^2 + b^2 = \pm\,2bzi\cot\beta \tag{14}$$

a family of spheres, whose circle of intersection is

$$x^2 + y^2 + b^2 = 0\,. \tag{15}$$

The family of cones of equation (2), when subjected to the transformation defined by equations (6) (7) (8) (11), becomes

$$(x^2 + y^2 + z^2 - b^2)^2 = \frac{4b^2(x^2 + y^2)}{\tan^2 \theta}. \tag{16}$$

For convenience later, a new parameter a will be introduced in place of θ through the equation

$$\tan^2 \theta + \tanh^2 a = 0. \tag{17}$$

Equation (16) is now

$$(x^2 + y^2 + z^2 - b^2)^2 = \frac{-4b^2(x^2 + y^2)}{\tanh^2 a}. \tag{18}$$

Equation (18) represents a family of anchor rings where $\dfrac{\pm bi}{\tanh a}$ is the distance from the centre of the family to the centre of any generating circle, and the radius of this circle is $\dfrac{\pm bi}{\sinh a}$.

It remains to determine the correspondence between the real and imaginary surfaces of the two systems.

To have real anchor rings the radii of the generating circles must be real and less than the respective radii of the rings.

These conditions will be fulfilled if a is taken as real and b as pure imaginary.

Making this assumption for b, equation (14) shows that, when $\cot \beta$ is real, the orthogonal spheres are real.

From equation (13) it follows that

$$\pm i \left[\frac{4b^2 r^2 + k^2}{4b^2 r^2 - k^2}\right] \text{ is to be real.}$$

It is found desirable later to introduce the condition

$$4b^2 + k^2 = 0, \tag{19}$$

and in that case r, which is the radius of any sphere of the cone system, is complex.

This condition does not restrict b since k may be chosen at pleasure. It is to be noticed that the value of b fixes the one constant element of the toroidal family, viz., the circle of intersection of the spheres. The assumptions regarding k and b make a pure imaginary.

From the discussion above it appears that the real rings of the toroidal system come from imaginary cones, and the real spheres come from imaginary spheres.

No values can be assigned to a, b, k such that the real surfaces of the cone system become the real surfaces of the toroidal system.

Having obtained a transformation which changes the cone system into a toroidal system, and having an expression for V (equation (5)) for the cone system, the next subject is the application of Lord Kelvin's method to the problem of obtaining an expression for V for the toroidal system.

The factor $\frac{1}{\rho}$ which appears in formula (1) is, from equation (8),

$$\frac{1}{\rho} = \frac{1}{\sqrt{x^2 + y^2 + (z + b)^2}}. \tag{20}$$

From equation (14)

$$\frac{1}{\sqrt{x^2 + y^2 + (z + b)^2}} = c \, . \sqrt{\frac{\sin \beta}{z}} (\cos \beta \pm i \sin \beta)^{\frac{1}{2}}. \tag{21}$$

From equations (14) and (18)

$$z = \frac{\pm bi \sin \beta}{\cosh \alpha \mp \cos \beta}. \tag{22}$$

From equations (21) and (22), equation (20) becomes

$$\frac{1}{\rho} = c_2 (\cosh \alpha \mp \cos \beta)^{\frac{1}{2}} (\cos \beta \pm i \sin \beta)^{\frac{1}{2}} \tag{23}$$

Equation (5) expresses V in terms of the parameters γ, φ, θ.

Let γ be replaced by β through equations (13) and (19), and let θ be replaced by α by means of equation (17). Equation (5) then becomes

$$V = [A_1 (\sin \beta \mp i \cos \beta)^m + B_1 (\sin \beta \mp i \cos \beta)^{-(m+1)}] \, .$$
$$[A_2 \cos n\varphi + B_2 \sin n\varphi] \, [A_3 P_m^n (\cosh \alpha) + B_3 Q_m^n (\cosh \alpha)] \, . \tag{24}$$

The expression for V for the toroidal system is now written at once by means of equations (23) and (24), giving

$$V = (\cosh \alpha \mp \cos \beta)^{\frac{1}{2}} [A_0 \cos (m + \tfrac{1}{2}) \beta + B_0 \sin (m + \tfrac{1}{2}) \beta] \, .$$
$$(A_2 \cos n\varphi + B_2 \sin n\varphi) \, [A_3 P_m^n (\cosh \alpha) + B_3 Q_m^n (\cosh \alpha)] \, . \tag{25}$$

Equation (25) is the solution of the problem previously stated.

The parameters α, β, φ, correspond to the surfaces of the toroidal system, and appear singly in each factor except the first.

The first factor is a function of both α and β but does not contain the accessory parameters m and n.

Equation (25) differs materially from the result ordinarily given (see

Byerly's Fourier Series and Spherical Harmonics, § 143). To obtain that form of solution it is necessary to consider the equation from which

$$[A_3 P_m^n (\cos\theta) + B_3 Q_m^n (\cos\theta)]$$

in equation (5) was derived, viz,

$$\frac{1}{\sin\theta} \cdot \frac{d}{d\theta}\left[\sin\theta \frac{d\Theta}{d\theta}\right] + \left[m(m+1) - \frac{n^2}{\sin^2\theta}\right]\Theta = 0 . \qquad (26)$$

In equation (34) the substitution is made

$$\Theta = L \,.\, (\sin\theta)^{-\frac{1}{2}} . \qquad (27)$$

The result is

$$\frac{d^2L}{d\theta^2} + \left\{\frac{\cos^2\theta}{4\sin^2\theta} + \tfrac{1}{2} + m(m+1) - \frac{n^2}{\sin^2\theta}\right\} L = 0 . \qquad (28)$$

A second substitution is then made

$$x = i\cot\theta . \qquad (29)$$

Equation (28) becomes

$$(1 - x^2)\frac{d^2L}{dx^2} - 2x\frac{dL}{dx} + \left[n^2 - \tfrac{1}{4} - \frac{(m+\frac{1}{2})^2}{1 - x^2}\right] L = 0 . \qquad (30)$$

The solution of equation (30) is

$$\begin{aligned} L &= CP_{n-\frac{1}{2}}^{m+\frac{1}{2}}(x) + DQ_{n-\frac{1}{2}}^{m+\frac{1}{2}}(x) \\ &= CP_{n-\frac{1}{2}}^{m+\frac{1}{2}}(i\cot\theta) + DQ_{n-\frac{1}{2}}^{m+\frac{1}{2}}(i\cot\theta) . \qquad (31) \end{aligned}$$

Equation (26) then has the solution

$$\Theta = (\sin\theta)^{-\frac{1}{2}}\,[CP_{n-\frac{1}{2}}^{m+\frac{1}{2}}(i\cot\theta) + DQ_{n-\frac{1}{2}}^{m+\frac{1}{2}}(i\cot\theta)] . \qquad (32)$$

From equation (17)

$$(\sin\theta)^{-\frac{1}{2}} = C_1(\sinh\alpha)^{-\frac{1}{2}} \qquad (33)$$

$$i \,.\, \cot\theta = \coth\alpha . \qquad (34)$$

The final form of the solution of equation (26) is

$$\Theta = C_1(\sinh\alpha)^{-\frac{1}{2}}\,[CP_{n-\frac{1}{2}}^{m+\frac{1}{2}}(\coth\alpha) + DQ_{n-\frac{1}{2}}^{m+\frac{1}{2}}(\coth\alpha)] . \qquad (35)$$

Using equation (35) in the reduction of equation (5) gives the following to replace equation (25):

$$V = C_2\frac{(\cosh\alpha \mp \cos\beta)^{\frac{1}{2}}}{(\sinh\alpha)^{\frac{1}{2}}}\,[A_2\cos n\varphi + B_2\sin n\varphi] .$$

$$[A_0\cos(m+\tfrac{1}{2})\beta + B_0\sin(m+\tfrac{1}{2})\beta][CP_{n-\frac{1}{2}}^{m+\frac{1}{2}}(\coth\alpha) + DQ_{n-\frac{1}{2}}^{m+\frac{1}{2}}(\coth\alpha)]. \qquad (36)$$

If R be defined as the distance of a point, (xyz), in the toroidal family, from the axis of symmetry, then from equations (14) and (18)

$$R = C_3 \frac{\sinh \alpha}{\cosh \alpha \mp \cos \beta}. \tag{37}$$

In equation (36) let the substitutions be made

$$m + \tfrac{1}{2} = p \tag{38}$$

$$n - \tfrac{1}{2} = q. \tag{39}$$

Equation (36) now assumes the more familiar form

$$V = \frac{1}{\sqrt{R}} [A \cos (q + \tfrac{1}{2}) \varphi + B \sin (q + \tfrac{1}{2}) \varphi] \cdot$$

$$[A_0 \cos p\beta + B_0 \sin p\beta] [CP_q^p (\coth \alpha) + DQ_q^p (\coth \alpha)]. \tag{40}$$

It is in this form that the result is ordinarily given.

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